

The Casimir effect as a candidate of dark energy

Jiro MATSUMOTO ^{*)}

Department of Physics, Nagoya University, Nagoya 464-8602, Japan

It is known that the simply evaluated value of the zero point energy of quantum fields is extremely deviated from the observed value of dark energy density. In this paper, we consider whether the Casimir energy, which is the zero point energy brought from boundary conditions, can cause the accelerated expansion of the Universe by using proper renormalization method and introducing the fermions of finite temperature living in $3+n+1$ space-time. We show that the zero temperature Casimir energy and the finite temperature Casimir energy can explain dark energy and dark matter, respectively.

§1. Introduction

The discovery of the accelerating expansion of the Universe in late 1990's¹⁾ has brought many models of dark energy in this ten and several years. Because the modifications of the Einstein equation are necessary to realize the accelerating expansion when we assume that the Universe is isotropic and homogeneous. Including the cosmological constant Λ into the Einstein equation is generally regarded as the most possible model to explain the accelerated expansion, because of its simplicity. This model, however, has a so-called fine-tuning problem. There are several ways to illustrate this problem, the fine tuning problem means that the energy scale of the cosmological constant Λ to explain the acceleration of the Universe is too small compared to the Planck scale M_{Pl} , which is the energy scale of the gravity. There have been attempts to explain the accelerating expansion of the Universe not by a cosmological constant but by dynamical fields without such a fine tuning. And there are also modified gravity models to explain such an acceleration. As a matter of fact, fine tunings of the free parameters are usually included in these models.

On the other hand, there are attempts to explain the observed value of the cosmological constant by the zero point energy of the quantum fields.²⁾ However, there would not have been persuasive explanations so far. In this paper, the zero point energy generated from boundary conditions, which is called the Casimir energy, is considered. To be concrete, the Casimir energy of the graviton and the fermions are considered when space-time has compact extra dimensions. Similar investigations are seen in the literature written by B. Greene and J. Levin.³⁾ The main difference of this paper from B. Greene and J. Levin's is to introduce finite temperature fermions instead of fermions and scalars at zero temperature. It will be shown later that reasonable models of cosmology cannot be constructed only from zero temperature fermions with periodic boundary conditions and the graviton, and the finite temperature effect can play a role of massive scalar fields.

In Sec. II, the finite temperature Casimir effect on $R^3 \times (S^1)^n$ space is calculated. And the cosmological influence from the Casimir effect is investigated in Sec. III by

^{*)} E-mail: matumoto@th.phys.nagoya-u.ac.jp

using the result from Sec. II. In Sec. IV, the influences to the experiments when we assume the fermions as neutrinos are shortly discussed. Concluding remarks are given in Sec. V. Units of $k_B = c = \hbar = 1$ are used and the gravitational constant $8\pi G$ is denoted by $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$ with the Planck mass of $M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19} \text{GeV}$.

§2. The finite temperature Casimir effect on $R^3 \times (S^1)^n$ space

In this section, we will calculate the finite temperature Casimir effect on $R^3 \times (S^1)^n$ space. We use the formulation and results in the literature written by S. Bellucci and A. A. Saharian.⁴⁾ The following calculations are the extensions of a part of the results in it⁴⁾ to use them in the case of finite temperature. And formulas written in *Integrals and Series*⁵⁾ will be used in the following without provisos.

The energy-momentum tensor of Dirac field on $R^3 \times S^n$ space is given as follows:

$$\langle 0|T_0^0|0\rangle = -\frac{N}{2d^n} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{l_n \in \mathbb{Z}^n} \omega_{k,l_n}, \quad (2.1)$$

$$\langle 0|T_i^i|0\rangle = \frac{N}{2d^n} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{l_n \in \mathbb{Z}^n} \frac{k_i^2}{\omega_{k,l_n}}, \quad (2.2)$$

where ω_{k,l_n} is defined as

$$\omega_{k,l_n} = \sqrt{m^2 + \mathbf{k}^2 + \sum_{i=4}^{n+3} \left(\frac{2\pi l_i}{d} \right)^2}. \quad (2.3)$$

In Eq. (2.2), T_i^i does not mean the summation with respect to i and $k_i \equiv 2\pi l_i/d$ for $i \geq 4$. The common periodic length d are used in Eq. (2.1) and (2.2) because the equivalence of each extra dimension has been assumed here. We denote the real degrees of freedom of particles by N . These Eqs. (2.1), (2.2) can be shown in the case of scalar field. So we assume Eqs. (2.1), (2.2) are also held in the case of the graviton. The graviton and the fermions are only considered as the degrees of freedom of particles in the following.

If we assume neutrinos as the fermions then $N = N_\nu \equiv 2^{\lfloor \frac{4+n}{2} \rfloor - 1}$ for each generation of neutrinos and $N = N_g \equiv -(n+4)(n+1)/2$ for the graviton as the degrees of freedom. Here, we assume that the neutrinos are Dirac type. The negativeness of the degrees of freedom for the graviton comes from the difference of statistics.

Then, we find the following relations from Eqs. (2.1) and (2.2):

$$p_0 = \frac{1}{3} \left\{ (n+1)\rho_0 - m \frac{\partial \rho_0}{\partial m} + d \frac{\partial \rho_0}{\partial d} \right\},$$

$$p_{b0} = -\rho_0 - \frac{d}{n} \frac{\partial \rho_0}{\partial d}, \quad (2.4)$$

where $\rho_0 = \langle 0|T_0^0|0\rangle$, $p_0 = -\sum_{i=1}^3 \langle 0|T_i^i|0\rangle/3$ and $p_{b0} = -\sum_{i=4}^{n+3} \langle 0|T_i^i|0\rangle/n$, because the signature of the metric is assumed to be $(+, -, -, -, -, \dots, -)$. Therefore, we do not need to calculate p_0 and p_{b0} , respectively. Calculations of ρ_0 will be omitted

and we only give the expression for ρ_0 because the expression of ρ_0 is given in former researches^{(4), (6), (7)} and we will conduct similar calculations to derive the finite temperature correction to the Casimir effect as follows,

$$\rho_0 = N \left(\frac{m}{2\pi} \right)^{\frac{4+n}{2}} \sum_{\substack{(l_4, l_5, \dots, l_{n+3}) \in \mathbb{Z}^n, \\ (l_4, l_5, \dots, l_{n+3}) \neq \mathbf{0}}} \frac{K_{\frac{4+n}{2}} \left[md \sqrt{\sum_{i=4}^{3+n} l_i^2} \right]}{d^{\frac{4+n}{2}} \sqrt{\sum_{i=4}^{3+n} l_i^2}^{\frac{4+n}{2}}}. \quad (2.5)$$

Here, $K_n(z)$ is the modified Bessel function of the second kind defined by

$$K_n(z) = \frac{\left(\frac{z}{2}\right)^n \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n + \frac{1}{2}\right)} \int_1^\infty e^{-zt} (t^2 - 1)^{\frac{2n-1}{2}} dt. \quad (2.6)$$

In the case of massless particles, Eq. (2.5) is simplified to

$$\rho_0 = N \frac{\Gamma\left(\frac{4+n}{2}\right)}{2\pi^{\frac{4+n}{2}} d^{4+n}} \sum_{\substack{(l_4, l_5, \dots, l_{n+3}) \in \mathbb{Z}^n, \\ (l_4, l_5, \dots, l_{n+3}) \neq \mathbf{0}}} \left(\sum_{i=4}^{3+n} l_i^2 \right)^{-\frac{4+n}{2}}. \quad (2.7)$$

By using the Matsubara formalism and the zeta function regularization, we obtain the following expression of the free energy:⁽⁸⁾

$$F_0 = E_0^{\text{ren}} + \Delta_T F_0, \quad (2.8)$$

$$\Delta_T F_0 \equiv k_B T \sum_J \ln \left(1 \mp e^{-\beta \omega_J} \right), \quad (2.9)$$

where J indicates the eigenvalue of the momentum and E_0^{ren} is the Casimir energy of zero temperature. In the second line of Eq. (2.9), the above sign of \mp is applied to bosons and the below sign of \mp is applied to fermions. The expression of $E_0^{\text{ren}} = V d^n \rho_0$ is given in Eq. (2.5) so that we consider the density of $\Delta_T F_0$:

$$\frac{\Delta_T F_0}{(V d^n)} = \frac{1}{d^n \beta} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \sum_{(l_4, l_5, \dots, l_{n+3}) \in \mathbb{Z}^n} \ln \left(1 \mp e^{-\beta \sqrt{m^2 + \mathbf{k}^2 + (2\pi/d)^2 \sum_{i=4}^{n+3} l_i^2}} \right), \quad (2.10)$$

where V is a volume of the three dimensional space. The energy density is, however, singular in the expression of Eq. (2.10). Therefore, we define the energy density ρ_T by subtracting the zero point energy without boundary conditions from Eq. (2.10) as defined in the Casimir energy of zero temperature:

$$\begin{aligned} \rho_T &= \frac{|N|}{d^n \beta} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \sum_{(l_4, l_5, \dots, l_{n+3}) \in \mathbb{Z}^n} \ln \left(1 \mp e^{-\beta \sqrt{m^2 + \mathbf{k}^2 + (2\pi/d)^2 \sum_{i=4}^{n+3} l_i^2}} \right) \\ &\quad - \frac{|N|}{\beta} \int \frac{d^{n+3} \mathbf{k}}{(2\pi)^{n+3}} \ln \left(1 \mp e^{-\beta \sqrt{m^2 + \mathbf{k}^2}} \right) \\ &= - \frac{|N|}{d^n \beta} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \sum_{s=1}^{\infty} \frac{(\pm 1)^s}{s} \left\{ \sum_{(l_4, l_5, \dots, l_{n+3}) \in \mathbb{Z}^n} e^{-s\beta \sqrt{m^2 + \mathbf{k}^2 + (2\pi/d)^2 \sum_{i=4}^{n+3} l_i^2}} \right\} \end{aligned}$$

$$\begin{aligned}
& - d^n \int \frac{d^n \mathbf{k}'}{(2\pi)^n} e^{-s\beta \sqrt{m^2 + \mathbf{k}^2 + \mathbf{k}'^2}} \Big\} \\
& = - \frac{|N|}{d^n \beta} \sum_{s=1}^{\infty} \frac{(\pm 1)^s}{s} \sum_{j=0}^{n-1} \Delta_j(s, d, \beta), \tag{2.11}
\end{aligned}$$

$$\begin{aligned}
\Delta_j(s, d, \beta) & \equiv d^j \int \frac{d^{3+j} \mathbf{k}}{(2\pi)^{3+j}} \sum_{(l_{5+j}, \dots, l_{n+3}) \in \mathbb{Z}^{n-j-1}} \left\{ \sum_{l_{4+j} \in \mathbb{Z}} e^{-s\beta \sqrt{m^2 + \mathbf{k}^2 + (2\pi/d)^2 \sum_{i=4+j}^{n+3} l_i^2}} \right. \\
& \left. - d \int_{-\infty}^{\infty} \frac{dk_{j+4}}{2\pi} e^{-s\beta \sqrt{m^2 + \mathbf{k}^2 + k_{j+4}^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2}} \right\}. \tag{2.12}
\end{aligned}$$

Eq. (2.12) can be transformed into

$$\begin{aligned}
\Delta_j(s, d, \beta) & = 4s\beta d^{j+1} \sum_{(l_{5+j}, \dots, l_{n+3}) \in \mathbb{Z}^{n-j-1}} \sum_{u=1}^{\infty} \left\{ \frac{\sqrt{m^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2}}{2\pi} \right\}^{\frac{5+j}{2}} \\
& \times \frac{K_{\frac{j+5}{2}} \left[\sqrt{m^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2} \sqrt{s^2 \beta^2 + u^2 d^2} \right]}{\left[\sqrt{s^2 \beta^2 + u^2 d^2} \right]^{\frac{j+5}{2}}}. \tag{2.13}
\end{aligned}$$

The derivation of Eq. (2.13) is written down in Appendix. Substituting Eq. (2.13) into Eq. (2.11), we obtain

$$\begin{aligned}
\rho_T & = -4|N| d^{j-n+1} \sum_{s=1}^{\infty} (\pm 1)^s \sum_{j=0}^{n-1} \sum_{(l_{5+j}, \dots, l_{n+3}) \in \mathbb{Z}^{n-j-1}} \sum_{u=1}^{\infty} \left\{ \frac{\sqrt{m^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2}}{2\pi} \right\}^{\frac{5+j}{2}} \\
& \times \frac{K_{\frac{5+j}{2}} \left[\sqrt{m^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2} \sqrt{s^2 \beta^2 + u^2 d^2} \right]}{\left[\sqrt{s^2 \beta^2 + u^2 d^2} \right]^{\frac{5+j}{2}}}. \tag{2.14}
\end{aligned}$$

Eq. (2.14) can be rewritten in more simple form as:

$$\rho_T = - \frac{2|N|m^{4+n}}{(2\pi)^{\frac{4+n}{2}}} \sum_{s=1}^{\infty} (\pm 1)^s \sum_{(l_4, \dots, l_{n+3}) \in \mathbb{Z}^n} \frac{K_{\frac{4+n}{2}} \left[m \sqrt{\beta^2 s^2 + d^2 \sum_{i=4}^{3+n} l_i^2} \right]}{m^{\frac{4+n}{2}} \sqrt{\beta^2 s^2 + d^2 \sum_{i=4}^{3+n} l_i^2}}. \tag{2.15}$$

The way of this transformation is identical with that in the appendix of S. Bellucci and A. A. Saharian.^{4) *} From Eqs. (2.5) and (2.15), we obtain the finite temperature

^{*}) To explain the way of the transformation briefly, making $\Delta_j(s, \beta, d)$ from Eq. (2.15) and comparing it to Eq. (2.13) gives the equivalence of Eqs. (2.14) and (2.15).

Casimir energy density $\rho_{\text{casimir}} = \rho_0 + \rho_T$ in the collected form:

$$\rho_{\text{casimir}} = -\frac{|N|m^{4+n}}{(2\pi)^{\frac{4+n}{2}}} \sum_{\substack{s \in \mathbb{Z}, \\ (l_4, \dots, l_{n+3}) \in \mathbb{Z}^n, \\ (s, l_4, \dots, l_{n+3}) \neq \mathbf{0}}} (1 - 2\delta_{s0})^\alpha (-1)^{\alpha s} \frac{K_{\frac{4+n}{2}} \left[m \sqrt{\beta^2 s^2 + d^2 \sum_{i=4}^{3+n} l_i^2} \right]}{m^{\frac{4+n}{2}} \sqrt{\beta^2 s^2 + d^2 \sum_{i=4}^{3+n} l_i^2}}, \quad (2.16)$$

where the notation $(-1)^\alpha = 1$ for bosons and $(-1)^\alpha = -1$ for fermions is used.

§3. Cosmic acceleration caused by the Casimir effect

3.1. Friedmann-Lemaître equations

First, we assume Friedmann-Lemaître-Robertson-Walker (FLRW) metric of the flat space,

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j + b^2(t) \delta_{\diamond \heartsuit} dx^\diamond dx^\heartsuit, \quad (3.1)$$

where $i, j = 1, 2, 3$ and $\diamond, \heartsuit = 4, 5, \dots, n+3$. And $\mu, \nu = 0, 1, 2, \dots, n+3$ will be used. Then, we can write down Friedmann-Lemaître (FL) equations as follows:

$$3H^2 + 3nHH_b + \frac{1}{2}n(n-1)H_b^2 = \kappa^2 d^n(t)\rho, \quad (3.2)$$

$$-3H^2 - 2\dot{H} - 2nHH_b - \frac{1}{2}n(n+1)H_b^2 - n\dot{H}_b = \kappa^2 d^n(t)p, \quad (3.3)$$

$$-6H^2 - 3\dot{H} + 3(1-n)HH_b - \frac{1}{2}n(n-1)H_b^2 + (1-n)\dot{H}_b = \kappa^2 d^n(t)p_b. \quad (3.4)$$

Here, $d(t) = d_0 b(t)$ is the periodic length of the compact dimensions and d_0 is the current value of it. The Hubble rate of the three dimensions is defined by $H(t) \equiv \dot{a}(t)/a(t)$ and the expansion rate of extra dimensions is defined by $H_b(t) \equiv \dot{b}(t)/b(t)$. ρ , p and p_b are defined as $\rho = -T_0^0$, $p = T_i^i/3$ and $p_b = T_\diamond^\diamond/n$, respectively. By eliminating the terms $\dot{H}(t)$ and $H^2(t)$ from Eq. (3.4), the equation of motion for the size of extra dimensions is derived:

$$\frac{1}{a^3 b^n} \frac{d}{dt} (a^3 b^n H_b) = \frac{\kappa^2}{n+2} d^n (\rho + 2p_b - 3p). \quad (3.5)$$

On the other hand, we obtain the equation of continuity $\nabla_\mu T^{\mu 0} = 0$ in the following form,

$$\dot{\rho} + 3H(\rho + p) + nH_b(\rho + p_b) = 0. \quad (3.6)$$

The Casimir energy density, pressure and the pressure in extra dimensions of zero temperature are given by Eqs. (2.4) and (2.5). Substituting Eq. (2.5) into Eq. (2.4) for p_0 yields the relation $p_0 = -\rho_0$, correctly. However, the total pressure from the Casimir effect have not been obtained. It can be seen from Eqs. (2.12) and (2.15)

that the Casimir energy density in the FLRW Universe is obtained by putting d and β into $d(t)$ and $\beta(t)$ in Eq. (2.15). The concrete form of the $\beta(t)$ will be given later. From the definition of the free energy, we obtain the following expressions of the pressures:

$$p_{\text{casimir}} = -\rho_{\text{casimir}} - \frac{a(t)}{3} \frac{\partial \rho_{\text{casimir}}}{\partial \beta} \frac{\partial \beta}{\partial a(t)}, \quad (3.7)$$

$$p_{b,\text{casimir}} = -\rho_{\text{casimir}} - \frac{b(t)}{n} \left(\frac{\partial \rho_{\text{casimir}}}{\partial b(t)} + \frac{\partial \rho_{\text{casimir}}}{\partial \beta} \frac{\partial \beta}{\partial b(t)} \right), \quad (3.8)$$

where ρ_{casimir} is treated as $\rho_{\text{casimir}}[b(t), \beta\{a(t), b(t)\}]$ instead of $\rho_{\text{casimir}}[d(t), \beta(t)]$. These Eqs. (3.7) and (3.8) are consistent with equations in (2.4) and Eq. (3.6), respectively.

Next question is the $a(t)$ and $b(t)$ dependence of the temperature. From the definition of the free energy, we have the entropy:

$$S_{\text{casimir}} \propto -\frac{\partial(\rho_{\text{casimir}} a^3 b^n)}{\partial T} \propto a^3 b^n \beta^2 \frac{\partial \rho_T}{\partial \beta}. \quad (3.9)$$

The adiabatic expansion of the Universe implies,

$$dS_{\text{casimir}} = 0. \quad (3.10)$$

Thus, if the term proportional to $\beta^{-4-n}(t)$ is a dominant component of ρ_T , then we have,

$$\beta(t) \propto a^{\frac{3}{3+n}}(t) b^{\frac{n}{3+n}}(t). \quad (3.11)$$

3.2. Expansion history of the Universe

It is difficult to solve Eqs. (3.2)–(3.4), but we can find the qualitative behavior of the solution. We can realize the following scenario if the values of parameters are properly selected.

As the Universe expands, the matter density decreases, and the right hand side of Eq. (3.5) becomes small and it can vanish at a certain time. Then the Universe evolves as to make the right hand side of Eq. (3.5) zero. To say more exactly, the scale factor $b(t)$ moves to satisfy $\dot{b}(t) \approx 0$ and $\rho + 2p_b - 3p \approx 0$. Because there is the stable solution that satisfies $\dot{b}(t) \approx 0$ and $\rho + 2p_b - 3p \approx 0$. It is understood from the behavior of the perturbation $\delta b(t) = b(t) - b_{\text{solution}}$. The condition $\delta b(t) > 0$ makes $\rho + 2p_b - 3p$ to be less than zero and yields $H_b(t) = \dot{b}(t)/b(t) < 0$, and $\delta b(t) < 0$ makes $\rho + 2p_b - 3p$ to be more than zero and yields $H_b(t) > 0$. If the conditions of this stable solution are satisfied, the time evolution of the typical four dimensional energy density of zero temperature, $b^{-4}(t)$, is faster than constant and more slowly than $a^{\frac{12}{n}}(t)$, and the evolution of the typical four dimensional energy density of finite temperature, $\beta^{-n-4}(t)b^n(t)$ is more slowly than $a^{-3\frac{4+n}{3+n}}(t)$ and faster than $a^{-3}(t)$. These width of the time dependence are caused from the difference between $\dot{b}(t) = 0$ and $\rho + 2p_b - 3p = 0$. Here, $(\rho + 2p_b - 3p)_{\text{casimir}} \sim \beta^{-4-n}$ is assumed and $\rho_{\text{matter}} \propto a^{-3}$ is considered. When $n \rightarrow \infty$, these conditions agree with each other and the solution

is determined completely. It is found from the behavior of four dimensional energy densities of the Casimir effect that the zero temperature part of the Casimir energy density act as dark energy and the finite temperature part of that act as dark matter in this scenario. Moreover, the negativeness of the Hubble rate of extra dimensions, which is caused by the trapping of the stable solution, contributes to the dark energy density, effectively.

We will check whether the solution satisfies such a stability when $n = 1$ as an example. The Casimir energy density is given as:

$$\begin{aligned} \rho_{\text{casimir}} = & -\frac{1}{4\pi^2} \sum_{\substack{s, l_4 \in \mathbb{Z}, \\ (s, l_4) \neq (0, 0)}} \left[\frac{15}{2} \frac{1}{(s^2\beta^2 + l_4^2 d^2)^{\frac{5}{2}}} \right. \\ & + \frac{N_\psi}{2} (1 - 2\delta_{s0}) (-1)^s \frac{m^2 e^{-m\sqrt{s^2\beta^2 + l_4^2 d^2}}}{(s^2\beta^2 + l_4^2 d^2)^{\frac{3}{2}}} \\ & \left. \times \left\{ 1 + \frac{3}{m\sqrt{s^2\beta^2 + l_4^2 d^2}} + \frac{3}{m^2 (s^2\beta^2 + l_4^2 d^2)} \right\} \right]. \end{aligned} \quad (3.12)$$

The first line of Eq. (3.12) are contributions from the graviton and the second line of Eq. (3.13) are contributions from massive fermions, respectively. The leading terms of the energy density are given by,

$$\begin{aligned} \rho_{\text{casimir}} \sim & -\frac{15}{4\pi^2} \left(\frac{1}{d^5} + \frac{1}{\beta^5} \right) \\ & + \frac{N_\psi}{4\pi^2} m^2 \left\{ e^{-md} \left(\frac{1}{d^3} + \frac{3}{md^4} + \frac{3}{m^2 d^5} \right) + e^{-m\beta} \left(\frac{1}{\beta^3} + \frac{3}{m\beta^4} + \frac{3}{m^2 \beta^5} \right) \right\}. \end{aligned} \quad (3.13)$$

Eq. (3.13) indicates that if $\beta \ll 1/m$ or $d \ll 1/m$ then $\rho_{\text{casimir}} > 0$, else if $\beta \gg 1/m$ and $d \gg 1/m$ then $\rho_{\text{casimir}} < 0$, in the case of $N_\psi > 5$. At the same time, the balancing condition, $\rho + 2p_b - 3p$, is rewritten by,

$$\begin{aligned} \rho + 2p_b - 3p = & 2\rho_{\text{casimir}} - 2d \frac{\partial \rho_{\text{casimir}}}{\partial d} + \frac{\beta}{4} \frac{\partial \rho_{\text{casimir}}}{\partial \beta} + \rho_{\text{matter}} \\ \sim & -\frac{15}{4\pi^2} \left(\frac{12}{d^5} + \frac{3}{4\beta^5} \right) + \frac{N_\psi}{4\pi^2} m^2 \left\{ e^{-md} \left(\frac{2m}{d^2} + \frac{14}{d^3} + \frac{36}{md^4} + \frac{36}{m^2 d^5} \right) \right. \\ & \left. + e^{-m\beta} \left(-\frac{m}{4\beta^2} + \frac{1}{2\beta^3} + \frac{9}{4m\beta^4} + \frac{9}{4m^2 \beta^5} \right) \right\} + \rho_{\text{matter}}. \end{aligned} \quad (3.14)$$

We find that if $d \gg \beta$ then $\rho + 2p_b - 3p$ can be zero when $\rho_{\text{casimir}} > 0$, else if $d \ll \beta$ then $\rho + 2p_b - 3p$ can be zero when $\rho_{\text{casimir}} < 0$ by combining Eqs. (3.13) and (3.14). Therefore, the condition, $d > \beta$, must be satisfied to be $\rho_{\text{casimir}} > 0$ when $\rho + 2p_b - 3p = 0$. Then, we find that the solution is stable because the Casimir components of the balancing condition, $\rho + 2p_b - 3p$, is negative in the case of $b \gg 1$, and $\rho + 2p_b - 3p$ is positive in the case of $b \ll 1$. If the finite temperature effect is not considered, then $\rho + 2p_b - 3p = 0$ is not realized when ρ_{casimir} is positive so that

dark energy cannot explained only from zero temperature fermions with periodic boundary conditions and the graviton.

We obtain the evolution equation of $a(t)$ by eliminating the terms proportional to $\dot{H}_b(t)$ and $H_b^2(t)$ from Eqs. (3.2)–(3.4),

$$\frac{1}{a^3 b^n} \frac{d}{dt} (a^3 b^n H) = \frac{\kappa^2}{2+n} d^n \{ \rho + (n-1)p - n p_b \}. \quad (3.15)$$

Combining this equation (3.15) and Eq. (3.5) gives,

$$\frac{1}{a^3 b^n} \frac{d}{dt} \{ a^3 b^n (H - H_b) \} = \kappa^2 d^n (p - p_b). \quad (3.16)$$

On the other hand, the temperature of neutrinos are derived from the entropy conservation when we assume neutrinos as the fermions which cause the Casimir effect:

$$T_{\nu 0} = \left(\frac{4}{11} \right)^{\frac{1}{3}} \left(\frac{a_0}{b_0} \right)^{\frac{n}{3+n}} T_{\gamma 0}, \quad (3.17)$$

where $T_{\gamma 0} = 2.725\text{K}$, the scale factors a and b are normalized at the time of decoupling, so that $a_{\text{dec}} = b_{\text{dec}}$. And we have only considered the leading terms of the entropy, therefore there appear correction terms proportional to the powers of mass. We can find that $T_{\nu 0}$ of this scenario can be larger than that of ΛCDM model, $T_{\nu 0, \Lambda} = 1.945\text{K}$. We need to be careful to the fact that the temperature given in Eq. (3.17) is not the temperature of three dimensional spaces but the effective temperature of $3+n$ dimensional spaces.

As the end of this section, an example of values of the parameters, m_1 , n , d and β is given when neutrinos are assumed as the fermions which cause the Casimir effect. The reason why neutrinos are considered is that the scale of masses of neutrinos makes the energy scale of the Casimir energy near to the observational value of dark energy. Here, m_1 is the mass of ν_1 and the masses of ν_2 and ν_3 are determined from $\Delta m_{\odot}^2 = 7.58 \times 10^{-5} \text{eV}^2$ and $|\Delta m_A^2| = 2.35 \times 10^{-3} \text{eV}^2$ ⁹⁾ by assuming normal hierarchy. The appropriate values of the parameters are $m_1 < 0.1 \text{meV}$, $n = 8$, $d = 66 \mu\text{m}$ and $T_{\nu 0} = 30\text{K}$. Then dark energy density and dark matter density can be explained by the Casimir energy density. Here, the temperature of the graviton is treated as $0.6/1.945 \times T_{\nu 0}$. However, the temperature of the graviton does not so much affect to the energy density, because the obtained energy density from the Casimir effect changes within 10% for the variation of temperature, $T_{g0} = (0 - 0.8)/1.945 \times T_{\nu 0}$.

In brief, the values of the parameters are determined in the following procedure. First, the practically free parameters are only m_1 and n in this model. Because d is determined by Eq. (3.5) if m_1 , n , and β are fixed, and β is almost determined by the generation of the balancing point, $\rho + 2p_b - 3p \approx 0$, and the condition, $\rho_{\text{casimir}} > 0$, on the balancing point. It is also noted that β is almost constant on the balancing point. On the other hand, m_1 and n are determined to fit the energy density from the Casimir effect on the balancing point to the values from observations. If $n < 8$ then there are no solutions to fit the cosmological observations in the range of $m_1 \in \mathbb{R}$.

For example, if $n = 6$, $m_1 < 0.1 \text{ meV}$, $d = 23.5 \mu\text{m}$ and $T_{\nu 0} = 80 \text{K}$, we obtain 20 times larger value of the energy density compared to the observed value of the critical density. Else if $n > 8$ except for $n = 9$ then m_1 , d , and β must to be large compared to those of $n = 8$ to fit the values from observations.

§4. Influences to the experiments and the observations

In this section, we consider the influences to the experiments and the observations when we assume neutrinos cause accelerating expansion of the Universe. First, this model has a continuous symmetry of translation for the three dimensional space and a discrete symmetry of spatial translation for extra compacted spaces, because the coupling to the $3 + 1$ dimensional matter with neutrinos breaks a continuous symmetry but the geometry $(S^1)^n$ keeps a discrete symmetry. Meanwhile, the momentum of the extra dimension is discretized by the same reason. Therefore, there is the momentum conservation, entirely. The momentum conservation guarantees,

$$m_{j,\text{eff}}^2 - m_{k,\text{eff}}^2 = m_j^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=4}^{n+3} l_i^2 - \left\{ m_k^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=4}^{n+3} l_i^2 \right\} = m_j^2 - m_k^2, \quad (4.1)$$

so that there is no influence to the neutrino oscillations if we consider the zero mode.

Next, let us consider the correction for the decay width of the Z boson. The effect of extra dimensions are expressed as corrections for masses, then the decay width of a Z boson into a neutrino pair is enhanced by the factor,

$$\sum_{(l_4, \dots, l_{n+3}) \in \mathbb{Z}^n} \left(1 - \frac{\left(\frac{2\pi}{d}\right)^2 \sum_{i=4}^{n+3} l_i^2}{M_Z^2 - m^2} \right) \sqrt{1 - 4 \frac{\left(\frac{2\pi}{d}\right)^2 \sum_{i=4}^{n+3} l_i^2}{M_Z^2 - 4m^2}}, \quad (4.2)$$

where M_Z is the mass of the Z boson, and m is a neutrino’s mass. If $(l_4, \dots, l_{n+3}) = \mathbf{0}$, then the factor (4.2) certainly becomes unity. Dividing the summation in (4.2) into two pieces, $M_Z^2 - 4m^2 > 4 \left(\frac{2\pi}{d}\right)^2 \sum_{i=4}^{n+3} l_i^2$ and $M_Z^2 - 4m^2 \leq 4 \left(\frac{2\pi}{d}\right)^2 \sum_{i=4}^{n+3} l_i^2$, then the former part of the summation gives a positive finite value which is larger than unity, and the latter part of the summation becomes zero because the terms $l_i = \mu$ and $l_i = -\mu$ are canceled by considering Eq. (A.3). Thus, neutrinos cannot be thought of the fermions which cause the Casimir effect because the decay width of the Z boson is terribly enhanced.

§5. Conclusions

It has been investigated whether or not the Casimir effect from the fermions and the graviton can explain dark energy if they are the only massless particles which can go through the compact extra dimensions. In the second section, the finite temperature Casimir effect has been calculated on $R^3 \times (S^1)^n$ space. And it has been seen that zero temperature Casimir energy and finite temperature Casimir energy can explain dark energy and dark matter, respectively, by considering the FL equations on $R^3 \times (S^1)^n \times R$ space-time in the third section. The behavior of the energy

come from the Casimir effect may resemble that of tracker solution in quintessence model.¹⁰⁾ However, neutrinos cannot be the fermions which cause accelerating expansion of the Universe, because the effect of extra dimension enhances the decay width of the Z boson.

The forth coming problem is understanding the time evolution of scale factors in the era, which corresponding to matter dominant era in Λ CDM model by assuming realistic initial conditions concretely. And it should be investigated that the effect to the matter density perturbation.

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Appendix

$\Delta_j(s, d, \beta)$ defined in Eq. (2.12) can be arranged by using the Abel-Plana formula,

$$\sum_{n=0}^{\infty} F(n) - \int_0^{\infty} dt F(t) = \frac{1}{2} F(0) + i \int_0^{\infty} \frac{dt}{e^{2\pi t} - 1} [F(it) - F(-it)], \quad (\text{A.1})$$

as follows,

$$\begin{aligned} \Delta_j(s, d, \beta) = & d^j \int \frac{d^{3+j} \mathbf{k}}{(2\pi)^{3+j}} \sum_{(l_{5+j}, \dots, l_{n+3}) \in \mathbb{Z}^{n-j-1}} \frac{2d}{\pi} \sqrt{m^2 + \mathbf{k}^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=5+j}^{n+3} l_i^2} \\ & \times \int_1^{\infty} \frac{dt}{e^{td\sqrt{m^2 + \mathbf{k}^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2}} - 1} \\ & \times \sin \left[s\beta \sqrt{m^2 + \mathbf{k}^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=5+j}^{n+3} l_i^2} \sqrt{t^2 - 1} \right], \quad (\text{A.2}) \end{aligned}$$

where the formula,

$$\begin{aligned} G_A^{(\alpha)}(it) - G_A^{(\alpha)}(-it) &= 2ie^{\alpha \ln(t^2 - A^2)} \sin[\pi\alpha] \theta(t - A), \\ G_A^{(\alpha)}(z) &\equiv \exp[\alpha \ln(A^2 + z^2)], \quad (\text{A.3}) \end{aligned}$$

has been used for $A = \frac{d}{2\pi} \sqrt{m^2 + \mathbf{k}^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2}$. The factor $1/(e^x - 1)$ is expanded with respect to e^{-x} as follows to integrate with respect to k and t :

$$\Delta_j(s, d, \beta) = d^j \int \frac{d^{3+j} \mathbf{k}}{(2\pi)^{3+j}} \sum_{(l_{5+j}, \dots, l_{n+3}) \in \mathbb{Z}^{n-j-1}} \frac{2d}{\pi} \sqrt{m^2 + \mathbf{k}^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=5+j}^{n+3} l_i^2}$$

$$\begin{aligned}
& \times \int_1^\infty dt \sum_{u=1}^\infty e^{-utd\sqrt{m^2 + \mathbf{k}^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2}} \\
& \times \sin \left[s\beta \sqrt{m^2 + \mathbf{k}^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=5+j}^{n+3} l_i^2} \right]. \tag{A.4}
\end{aligned}$$

Furthermore, the expansion of sinusoidal function with respect to its argument enables us to examine t integration:

$$\begin{aligned}
\Delta_j(s, d, \beta) &= -d^j \int \frac{d^{3+j}\mathbf{k}}{(2\pi)^{3+j}} \sum_{(l_{5+j}, \dots, l_{n+3}) \in \mathbb{Z}^{n-j-1}} \sum_{u=1}^\infty \sum_{v=1}^\infty \\
&\times \frac{2d}{\pi^{3/2}s\beta} \left(-\frac{2s^2\beta^2}{ud} \right)^v \frac{\Gamma(v+1/2)}{(2v-1)!} \left\{ m^2 + \mathbf{k}^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=5+j}^{n+3} l_i^2 \right\}^{\frac{v}{2}} \\
&\times K_v \left[ud \sqrt{m^2 + \mathbf{k}^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=5+j}^{n+3} l_i^2} \right], \tag{A.5}
\end{aligned}$$

where $K_\nu(z)$ is the function called the modified Bessel function or Macdonald function defined in Eq. (2.6). By transforming the Cartesian coordinate into the spherical coordinate in k integration gives,

$$\begin{aligned}
\Delta_j(s, d, \beta) &= -d^j \sum_{(l_{5+j}, \dots, l_{n+3}) \in \mathbb{Z}^{n-j-1}} \sum_{u=1}^\infty \sum_{v=1}^\infty \frac{2d}{\pi^{3/2}s\beta} \left\{ \frac{\sqrt{m^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2}}{2\pi ud} \right\}^{\frac{3+j}{2}} \\
&\times \left(-\frac{2s^2\beta^2}{ud} \right)^v \frac{\Gamma(v+1/2)}{(2v-1)!} \left\{ m^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=5+j}^{n+3} l_i^2 \right\}^{\frac{v}{2}} \\
&\times K_{v+\frac{j+3}{2}} \left[ud \sqrt{m^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=5+j}^{n+3} l_i^2} \right] \\
&= -d^j \sum_{(l_{5+j}, \dots, l_{n+3}) \in \mathbb{Z}^{n-j-1}} \sum_{u=1}^\infty \frac{2d}{\pi^{3/2}s\beta} \left\{ \frac{\sqrt{m^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2}}{2\pi ud} \right\}^{\frac{3+j}{2}} \\
&\times \frac{\partial}{\partial \ln s} \sum_{v=1}^\infty \pi^{\frac{1}{2}} (ud)^{\frac{j+3}{2}} \frac{1}{v!} (s^2\beta^2)^v \left\{ m^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=5+j}^{n+3} l_i^2 \right\}^{\frac{v}{2} + \frac{j+3}{4}} \\
&\times \frac{d^v}{d^v \zeta} \left\{ \zeta^{-\frac{j+3}{4}} K_{\frac{j+3}{2}}(\zeta^{\frac{1}{2}}) \right\} \Big|_{\zeta=u^2 d^2 \left\{ m^2 + \left(\frac{2\pi}{d}\right)^2 \sum_{i=5+j}^{n+3} l_i^2 \right\}}
\end{aligned}$$

$$\begin{aligned}
&= d^j \sum_{(l_{5+j}, \dots, l_{n+3}) \in \mathbb{Z}^{n-j-1}} \sum_{u=1}^{\infty} \frac{2d}{\pi^{3/2} s \beta} \left\{ \frac{\sqrt{m^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2}}{2\pi u d} \right\}^{\frac{3+j}{2}} \\
&\quad \times \pi^{\frac{1}{2}} (ud)^{\frac{j+3}{2}} \left\{ m^2 + \left(\frac{2\pi}{d} \right)^2 \sum_{i=5+j}^{n+3} l_i^2 \right\}^{\frac{j+3}{4}} \\
&\quad \times \frac{\partial}{\partial \ln s} \left\{ \frac{K_{\frac{j+3}{2}} \left[\sqrt{m^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2} \sqrt{s^2 \beta^2 + u^2 d^2} \right]}{\left[\sqrt{m^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2} \sqrt{s^2 \beta^2 + u^2 d^2} \right]^{\frac{j+3}{2}}} \right\} \\
&= 4s\beta d^{j+1} \sum_{(l_{5+j}, \dots, l_{n+3}) \in \mathbb{Z}^{n-j-1}} \sum_{u=1}^{\infty} \left\{ \frac{\sqrt{m^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2}}{2\pi} \right\}^{\frac{5+j}{2}} \\
&\quad \times \frac{K_{\frac{j+5}{2}} \left[\sqrt{m^2 + (2\pi/d)^2 \sum_{i=5+j}^{n+3} l_i^2} \sqrt{s^2 \beta^2 + u^2 d^2} \right]}{\left[\sqrt{s^2 \beta^2 + u^2 d^2} \right]^{\frac{j+5}{2}}}. \tag{A.6}
\end{aligned}$$

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